XX. *On Mr.* SPOTTISWOODE'S *Contact Problems.*

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THE present communication consists of two parts.

The first part treats of the contact of conics with a given surface at a given point; this class of questions was first treated by Mr. SPOTTISWOODE in his paper "On the Contact of Conics with Surfaces," and general formulæ applicable to all such questions were given.

The results of that paper are here reproduced with some additions; with the exception of a few collateral theorems, these are all contained in the following Table:—

*Number of five-point conics through fixed point	= 6
*Order of surface formed by five-point conics through fixed axis .	= 8
Number of six-point conics through fixed axis	= 9†
*Number of seven-point conics	=70

The second part treats of the contact of a quadric surface with a surface of the order n; and in particular it determines the number of points at which a quadric (other than the tangent plane reckoned twice) can have four-branch contact with the surface. In his paper "On the Contact of Surfaces," Mr. SPOTTISWOODE proves that at an arbitrary point on a surface there is no other solution than the doubled tangent plane, and gives the conditions that must be satisfied by those points at which another solution is possible.

The method here adopted is an extension of that applied by JOACHIMSTAL to the contact of lines with curves and surfaces. The coordinates of a point on a conic are expressed in terms of a single parameter, those of a point on a quadric by two parameters. To determine the intersection with a given surface we have an equation in the parameter or parameters, and the conditions of contact are expressed in terms of the coefficients of that equation. The special case of the intersection of a quadric with a cubic surface is treated by the method of representation on a plane.

* These results constitute the additions.

† [Note by W. SPOTTISWOODE.—In the Memoir quoted by Professor CLIFFORD, it was stated that the number of conics passing through a given axis and having six-pointic contact with a surface at a given point is ten. In making this statement I overlooked the fact that, in order to put in evidence that a certain quantity was a factor of the equation which determines the positions of the planes of the conics, the equation was multiplied by a quantity D which is a linear function of the position. In reckoning the degree of the equation this factor must of course be discarded. The degree is consequently less by unity than that stated in the Memoir; viz. it is 9, as proved by Professor CLIFFORD.—July 3, 1873.]

PART I.—THE CONTACT OF CONICS WITH SURFACES OF ORDER n.

I.

The current plane-coordinates being denoted by

$$X_1, X_2, X_3, X_4,$$

let the equations of the three points A, B, C be respectively

$$0=a_1X_1+a_2X_2+a_3X_3+a_4X_4=\Sigma aX,$$
$$0=\Sigma bX,$$
$$0=\Sigma cX.$$

The quantities a_i, b_i, c_i ($i=1, 2, 3, 4$) are the coordinates of the points A, B, C. The symbol A itself I shall use indifferently, as denoting either the form ΣaX or the differential operator

$$a_1\partial_{x_1}+a_2\partial_{x_2}+a_3\partial_{x_3}+a_4\partial_{x_4}=\Sigma a\partial_x,$$

where x_1, x_2, x_3, x_4 are the current point-coordinates. It will be seen in the sequel that this double meaning is useful, while it does not introduce any confusion. Similar interpretations are of course to be given to the symbols B, C, and the like.

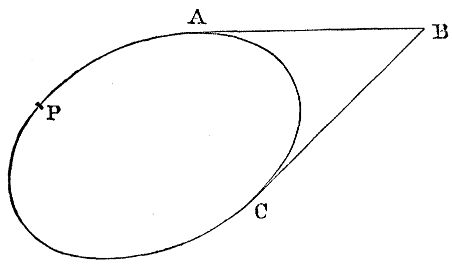
Consider now the point

$$P=A+\theta B+\theta^2C,$$

whose coordinates are $a_i+\theta b_i+\theta^2c_i (i=1, 2, 3, 4)$. If we suppose θ to take all possible values, the point P will describe a conic section whose tangential equation is

$$0=4AC-B^2=K_2.$$

To the value $\theta=0$ corresponds the point A, to $\theta=\infty$ the point C; while the equation shows that B is the intersection of the tangents at A and C.



To find the point at which this conic intersects a given surface u_n of the order n, we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation in θ of the order $2n$, the solution of which will give the values of θ belonging to the $2n$ points of intersection.

If in this equation the term independent of θ vanishes, then $\theta=0$ is a root of the equation; consequently the point A is one of the intersections, or the surface u_n passes through the point A. If also the coefficient of θ vanishes, another root of the equation coincides with zero, and *two* points of intersection are at A. And generally if the coefficient of θ^m is the first that does not vanish, m roots of the equation coincide with zero, and m points of intersection are united at A.

The result of substituting the coordinates of any point P in u_n may be conveniently represented by means of the differential operator P. It is known, in fact, that

$$(p_1\partial_{x_1}+p_2\partial_{x_2}+p_3\partial_{x_3}+p_4\partial_{x_4})^n\,.\,(x_1, x_2, x_3, x_4)^n = \underline{|n}\,.\,(p_1, p_2, p_3, p_4)^n;$$

or, which is the same thing, $P^n u_n$ is $\underline{|n}$ times the result of substituting the coordinates of P in u_n. Our result may therefore be stated in the following form:—

The necessary and sufficient conditions that the conic $K_2 = 4AC - B^2$ *may have* m*-point contact with the surface* u_n *at the point* A, *are that in the expansion of* $(A+\theta B+\theta^2 C)^n\,.\,u_n$ *in powers of* θ, *the* m*th power of* θ *is the lowest whose coefficient does not vanish.*

II.

Equating to zero the coefficients of 1, θ, θ^2 in this expansion, we obtain

$$0 = A^n\,.\,u_n,$$
$$0 = nA^{n-1}B\,.\,u_n,$$
$$0 = \tfrac{1}{2}n(n-1)A^{n-2}B^2\,.\,u_n + nA^{n-1}C\,.\,u_n.$$

Before proceeding further with these equations, it is convenient to make the following remarks upon their nature, which will serve to simplify the expression of them.

In the first place, then, we have here a series of relations among the coordinates of the points A, B, C and the coefficients of the surface u_n; and the determination of the coordinates and coefficients so as to satisfy a certain number of the relations presents us with the solution of various geometrical problems. These problems fall naturally into three classes.

1. The surface u_n and the point A are given. In this case the unknowns are the ratios of the eight quantities b, c, a singly infinite number of solutions corresponding to each conic; and we are accordingly able to satisfy seven of the equations*. The problem here is to find the number of conics which have seven-point contact at a given point of a given surface.

We may, however, impose beforehand certain restrictions upon the values of the unknowns, and so consider problems which involve a less number of the equations. While the number of the septactic conics is definite, the sextactic conics form a singly infinite series; and we may ask what is the number of them:

(*a*) whose planes pass through a given point,
(*b*) which meet a given line, or
(*c*) which touch a given plane.

The quintactic conics, again, form a doubly infinite system, and we may inquire about the number of them which satisfy two conditions; *e. g.* which pass through a given point.

* Viz. six besides the first, which is satisfied identically.

2. The surface u_n is given, but not the point A. In this case, as the point A is only restricted to be a point on the surface, we have two more unknowns, making nine in all. The problems here are, to find the order of the curve on the surface at every point of which there is a conic having eight-point contact, and to find the number of points at which there is a conic having nine-point contact.

As before, however, there are certain derived problems coming under this head which involve a less number of equations. We may seek the order of the curve traced out by points of contact of septactic conics satisfying one condition, sextactic satisfying two, &c.; or we may seek the number of septactic conics satisfying two conditions, sextactic satisfying three, &c.

3. The surface is not wholly given. We may here assign a number of relations sufficient to eliminate the quantities a, b, c, leaving one or more relations among the coefficients of u_n. The problems here are such as:—to find the number of surfaces in a pencil $u_n+\lambda v_n$ which admit of ten-point contact with some conic, or one of whose nine-point conics meets a given line.

In the present communication only problems of class 1 will be considered; the formulæ in this case may be very considerably simplified. The quantities a and the coefficients of u_n, then, are data of the problem; so that the first of our equations, $A^n u_n=0$, is satisfied by hypothesis. The next equation, $A^{n-1}Bu_n=0$, signifies that the point B lies in the tangent plane at A, as it obviously must if the conic touch the surface at A. We shall suppose this also to be satisfied from the commencement; that is, we shall regard B as a point moving in the tangent plane, and to be determined by construction in that plane. This may be effected analytically if we substitute for B, $\lambda A+\mu B+\nu B'$, where now B, B′ are regarded as fixed points in the tangent plane, and the three unknowns λ, μ, ν take the place of the four quantities b. There is, however, as will be seen, no occasion to make the substitution explicitly.

This being so, any relation involving B only beside the data must be regarded as the equation of a curve in the tangent plane. For example, $A^{n-2}B^2u_n=0$, expressing that B lies on the quadric polar of A, is the equation of the two chief tangents at that point. Generally, $B^n u_n=0$ is the equation of the intersection with u_n of the tangent plane; and the curves $AB^{n-1}u_n=0$, $A^2B^{n-2}u_n=0$, &c. are the successive polars of A in regard to that intersection.

The terms entering into our equations are of the general form

$$\theta^{p+2q}.\frac{\lfloor n}{\lfloor n-p-q\,\lfloor p\,\lfloor q}\,A^{n-p-q}B^pC^q.u_n.$$

Any term, therefore, is completely determined by the two numbers p and q, and might, for any thing that has yet appeared, be denoted by (p, q). In view, however, of subsequent substitutions, we shall keep in evidence the manner in which B and C are involved, and denote the term in question by the symbol $\theta^{p+2q}\,(B^pC^q)$.

As we do not consider any higher than seven-point contact, we have only the five equations:—

$$0=(B^2)+(C). \qquad (3)$$

$$0=(B^3)+(BC). \qquad (4)$$

$$0=(B^4)+(B^2C)+(C^2) \qquad (5)$$

$$0=(B^5)+(B^3C)+(BC^2). \qquad (6)$$

$$0=(B^6)+(B^4C)+(B^2C^2)+(C^3). \qquad (7)$$

III. *Conics through a fixed point.*

Combining equation (3) successively with (4) and (5), so as to obtain results homogeneous in B, we find

$$0=(B^3)(C)-(BC)(B^2),$$

$$0=(B^4)(C)^2-(B^2C)(B^2)(C)+(C^2)(B^2)^2.$$

If we regard C as a fixed point, these are equations of a cubic and a quartic curve in the tangent plane, on each of which B must lie if the conic K_2 has five-point contact. But these curves have a common node at A, and common tangents at it; for every term in each has at least one factor of the form (B^m), which we know to represent a polar of A in regard to the intersection of u_n by the tangent plane—that is, a curve touched by the chief tangents at A. Of their 12 intersections, then, 6 coincide with the point A; and there remain *six conics having five-point contact at* A *which pass through an arbitrary point* C.

IV. *Conics meeting a fixed axis through* A.

If the point C, instead of being altogether given, is movable on a fixed straight line through A, we may represent it by $A+\lambda C$; where now C is really a fixed point, and λ a quantity to be determined. When we make this substitution in our equations, they become*

$$0=(B^2)+\lambda(C),$$

$$0=(B^3)+\lambda(BC),$$

$$0=(B^4)+(B^2)+\lambda(B^2C)+2\lambda(C)+\lambda^2(C^2),$$

$$0=(B^5)+(B^3)+\lambda(B^3C)+2\lambda(BC)+\lambda^2(BC^2),$$

$$0=(B^6)+(B^4)+\lambda(B^4C)+(B^2)+2\lambda(B^2C)+\lambda^2(B^2C^2)+3\lambda(C)+3\lambda^2(C^2)+\lambda^3(C^3).$$

The last three admit of obvious simplifications by aid of the previous ones; and the system may finally be written

$$0=(B^2)+\lambda(C). \qquad (3')$$

$$0=(B^3)+\lambda(BC). \qquad (4')$$

$$(B^2)=(B^4)+\lambda(B^2C)+\lambda^2(C^2). \qquad (5')$$

$$(B^3)=(B^5)+\lambda(B^3C)+\lambda^2(BC^2). \qquad (6')$$

$$(B^4)-\lambda^2(C^2)=(B^6)+\lambda(B^4C)+\lambda^2(B^2C^2)+\lambda^3(C^3). \qquad (7')$$

* It must be remembered that (B) and terms involving A only vanish by hypothesis. The formulæ have also been simplified by the omission of certain coefficients depending on n which do not affect the final results.

Locus of poles of axis in regard to four-point conics.

If we select any plane through the line AC, there will be a singly infinite number of conics in the plane having four-point contact with the surface at A. The line AC will, as is well known, have the same pole in regard to all these conics—that is to say, the point B will be the same for the whole system. If we now allow the plane to turn round the axis, the point B will trace out a curve in the tangent plane. The equation to this curve is got by eliminating λ between equations (3′) and (4′); namely, it is

$$0=(B^2)(BC)-(B^3)(C). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4'')$$

We see, therefore, that *the locus of the poles of an axis in regard to all the four-point conics whose planes pass through it is a cubic curve in the tangent plane touching the chief tangents at* A, *which point is therefore a node on the curve.*

We might have inferred this from the fact that on any line through A there is only one point B, while this point coincides with A in the case of the two chief tangents; since at a point of inflexion all the four-point conics contain the inflexional tangent.

Number of six-point conics through an axis.

We have now to determine B so that the equations (3′), (4′), (5′), (6′) may be simultaneously satisfied. We know already that B must lie on the cubic (4″); it is necessary therefore to find some other locus on which it has to lie. First of all, then, we must eliminate λ between (3′) and (5′) and between (3′) and (6′); the results are,

$$(B^2)(C)^2=(B^4)(C)^2-(B^2C)(B^2)(C)+(B^2)^2(C^2)\equiv p_4, \text{ say};$$
$$(B^3)(C)^2=(B^5)(C)^2-(B^3C)(B^2)(C)+(B^2)^2(BC^2)\equiv q_5, \text{ say}.$$

Here $p_4=0$ and $q_5=0$ are curves touching the chief tangents at A, and of the degrees four and five respectively. But the equations are not homogeneous; in fact only the ratios and not the absolute values of the quantities a were determined by the fixing of the point A, and they may be regarded as involving an arbitrary factor whose square affects the left-hand side of the equations. It is, however, at once eliminated, and we obtain the homogeneous result,

$$(B^2)\,.\,q_5=(B^3)\,.\,p_4.$$

This is a curve of order 7 having two branches in each of the chief directions at A. Of its 21 intersections with the cubic (4″), then, 12 coincide with A, and there remain *nine positions of* B *which give sextactic conics through the fixed axis*; or we may say, *of the sextactic conics at the point* A, *there are nine whose planes pass through an arbitrary point* C.

System of five-point conics through an axis.

Since there is one five-point conic in every plane, if we consider all the planes through a fixed axis we shall obtain a singly infinite number of five-point conics. Of this system there is only one conic whose plane passes through an arbitrary point, viz. the conic determined by the plane through that point and the axis.

There are eight conics of the system which meet an arbitrary line.

The number of conics which meet an arbitrary line is clearly the same as the order of the surface which they trace out. Now, since through every point on the axis can be drawn six conics of the system (as proved in the last section), the axis is a six-fold line on the surface. The section of the surface, then, by a plane through the axis is made up of the axis taken six times over and the conic in that plane; or it is of the order *eight.* Q.E.D.

V. *Conics not subject to any condition.*

In order to get rid of the restriction of meeting a fixed axis, we must again modify our fundamental equations. We have to put them into a form in which they will represent *any* conic touching the surface u_n at the point A. For this purpose it is necessary and sufficient that C should be movable over a plane passing through A; since every conic through A must cut the plane in one other point, but this may be any point of the plane. We attain this analytically by substituting for λC in the second set of equations, $\lambda C+\mu D$, where C and D are now fixed points not in the tangent plane and not in any straight line through A. This is equivalent to still considering the conics which meet a given axis, but allowing that axis to move over a fixed plane.

The transformed equations are:—

$$0=(B^2)+\lambda(C)+\mu(D). \quad \text{(3''')}$$

$$0=(B^3)+\lambda(BC)+\mu(BD). \quad \text{(4''')}$$

$$(B^2)=(B^4)+\lambda(B^2C)+\mu(B^2D)+\lambda^2(C^2)+2\lambda\mu(CD)+\mu^2(D^2). \quad \text{(5''')}$$

$$(B^3)=(B^5)+\lambda(B^3C)+\mu(B^3D)+\lambda^2(BC^2)+2\lambda\mu(BCD)+\mu^2(BD^2). \quad \text{(6''')}$$

$$\left.\begin{aligned}(B^4)-\lambda^2(C^2)-2\lambda\mu(CD)-\mu^2(D^2)=(B^6)+\lambda(B^4C)+\mu(B^4D)+\lambda^2(B^2C^2)+2\lambda\mu(B^2CD)\\ +\mu^2(B^2D^2)+\lambda^3(C^3)+3\lambda^2\mu(C^2D)+3\lambda\mu^2(CD^2)+\mu^3(D^3).\end{aligned}\right\} \quad \text{(7''')}$$

Locus of poles of axis in given plane in regard to sextactics.

From the first two of these equations we obtain

$$1:\lambda:\mu=(BD)(C)-(BC)(D):(B^3)(D)-(B^2)(BD):(B^3)(C)-(B^2)(BC)$$
$$=n_1:l_3:m_3,\ \text{say};$$

here $n_1=0$ is the equation to a straight line passing through A, while $l_3=0$, $m_3=0$ are cubics touching the chief tangents at A.

Substituting these values in (5''') and (6'''), we obtain

$$n_1^2(B^2)=n_1^2(B^4)+n_1l_3(B^2C)+n_1m_3(B^2D)+l_3^2(C^2)+2l_3m_3(CD)+m_3^2(D^2)$$
$$=u_6, \text{ say;}$$
$$n_1^2(B^3)=n_1^2(B^5)+n_1l_3(B^3C)+n_1m_3(B^3D)+l_3^2(BC^2)+2l_3m_3(BCD)+m_3^2(BD^2)$$
$$=v_7, \text{ say.}$$

Here the curves $u_6=0$, $v_7=0$ are of the sixth and seventh orders respectively, each of them having *one* branch at A in each of the chief directions, and one other branch different for the two curves.

The equations

$$n_1^2(B^2)=u_6,$$
$$n_1^2(B^3)=v_7$$

must hold for six-point contact, but they are not homogeneous. Eliminating n_1^2, however, there results

$$(B^2).v_7=(B^3).u_6,$$

a curve of the ninth order, locus of the poles in regard to the sextactic conics of an axis moving in a fixed plane. This curve has *two* branches at A in each of the chief directions, and *one* other branch; and consequently is met by the plane ACD in *five* points coinciding with A, and in *four* other points. Now the plane ACD does not in general contain a sextactic conic; the pole of the axis can therefore only be in this plane when the axis itself is in the tangent plane. In this case there is a certain number of proper sextactic conics in planes through the axis, and it is clear that the pole of the axis in regard to any such conic is the point A. These conics, therefore, correspond to the *five* intersections of the plane ACD with the locus of poles which coincide with A; or *through an axis in the tangent plane can be drawn five proper sextactic conics.* In the tangent plane itself there are four improper conics having six-point contact; viz. the pair of chief tangents, which (as a sharp conic or line-pair reckoned among conics given tangentially) counts for two, and each chief tangent doubled.

Number of septactic conics.

There is a finite number of septactic conics at the point A; each of these meets the plane ACD in a determinate point $A+\lambda C+\mu D$, and fixes thereby a position of the point B. These positions of the point B must necessarily lie in the 9thic locus just investigated; it remains only to find a homogeneous relation which shall determine another locus for B, and to count the number of their intersections.

To this end we must first substitute for $1:\lambda:\mu$ their values $n_1:l_3:m_3$ in equation (7‴). The result is

$$n_1^3(B^4)-n_1l_3^2(C^2)-2n_1l_3m_3(CD)-n_1m_3^2(D^2)=n_1^3(B^6)+n_1^2l_3(B^4C)+n_1m_3(B^1D)$$
$$+n_1l_3^2(B^2C^2)+2n_1l_3m_3(B^2CD)+n_1m_3^2(B^2D)$$
$$+l_2^3(C^3)+3l_3^2m_3(C^2D)+3l_3m_3^2(CD^2)+m_3^3(D^3),$$

or

$$n_1t_5=w_9, \text{ say.}$$

The curve w_9 has *one* branch at A in each of the chief directions and one other branch. The curve t_6 has one branch in each of the chief directions and *two* other branches.

The equations (5), (6), (7) have now become

$$n_1^2(\mathrm{B}^2)=u_6,$$
$$n_1^2(\mathrm{B}^3)=v_7,$$
$$n_1 t_6=w_9.$$

The first two of these give us the curve already considered,

$$(\mathrm{B}^2)\,.\,v_7=(\mathrm{B}^3)\,.\,u_7,$$

which has at A two branches in the chief directions and one other. The first and third give

$$n_1\,.\,(\mathrm{B}^2)\,.\,w_9\,t_6 u_6,$$

a curve of order 12, having two branches in each of the chief directions and two other branches. Of the 108 intersections of these curves, then, $24+8+4+2=38$ coincide with A, leaving 70 for the number of septactic conics.

PART II.—THE CONTACT OF QUADRIC SURFACES WITH SURFACES OF ORDER n.

I. *Conditions of contact.*

Let A, B, C, D be four points forming a tetrahedron, whose tangential equations are

$$0=\Sigma a\mathrm{X},\ \Sigma b\mathrm{X},\ \Sigma c\mathrm{X},\ \Sigma d\mathrm{X}$$

respectively, their coordinates being a_i, b_i, c_i, d_i $(i=1, 2, 3, 4)$. Then the point

$$\mathrm{P}\equiv\mathrm{A}+\theta\mathrm{B}+\phi\mathrm{C}+\theta\phi\mathrm{D},$$

whose coordinates are

$$p_i\equiv a_i+\theta b_i+\phi c_i+\theta\phi d_i\ (i=1, 2, 3, 4),$$

will, if we suppose θ and ϕ to take all possible values, trace out a quadric surface whose tangential equation is

$$0=\mathrm{AD}-\mathrm{BC}\equiv\mathrm{Q}_2.$$

To the pair of values

$\theta=0$,	$\phi=0$,	corresponds the point A,
$\theta=\infty$,	$\phi=0$,	„ „ B,
$\theta=0$,	$\phi=\infty$,	„ „ C,
$\theta=\infty$,	$\phi=\infty$,	„ „ D.

The equation shows that AB, AC, BD, CD are generating lines.

If, now, we wish to find the nature of the curve in which this quadric intersects a given surface u_n of the order n, we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation which is of the order n in θ and in ϕ separately. If we regard θ and ϕ as coordinates of a point on the quadric surface, the equation just found is that of the curve of intersection.

Suppose that in this equation the term independent of θ and ϕ vanishes, then the equation is satisfied by the pair of values $\theta=0$, $\phi=0$, or the curve of intersection passes through the point A. Now the various directions in which we may start from the point A are determined by the initial value of the ratio $\theta:\phi$ when we move along them. The direction in which the intersection-curve starts from A is therefore that obtained by neglecting in the equation terms of higher order than the first; and we see that there is only one such direction.

If, however, not only the constant term but the coefficients of θ and ϕ in the equation vanish, the initial directions are obtained by equating to zero the terms of the second order, *i. e.* by neglecting in the equation all terms of higher order than the second. In this case, then, there are two such directions, the intersection-curve has a double point at A, and the quadric has with the surface u_n an ordinary or *two-branch* contact.

Again, if the coefficients of the terms of the third order are the first that do not vanish, the initial directions are obtained by neglecting all the terms of higher order, and there are consequently three of them. Thus the intersection-curve has a *triple* point at A, and the two surfaces have a *three-branch* contact.

And so generally, if the coefficients of the terms of the mth order are the first that do not vanish, the intersection-curve has at A a multiple point of order m, and the two surfaces have at that point an m-branch contact.

The result of these considerations may be stated as follows:—

The necessary and sufficient conditions that the quadric $Q_2\equiv AD-BC$ *may have* m*-branch contact with the surface* u_n *at the point* A, *are that in the expansion of* $(A+\theta B+\phi C+\theta\phi D)^n.u_n$ *in powers and products of* θ *and* ϕ, *the terms of order* m *in* θ *and* ϕ *are the lowest whose coefficients do not vanish.*

II. *Quadrics of four-branch contact.*

The equations we shall have to employ are so simple in form that it is unnecessary to employ the abridged notation of the former Part. We shall merely omit the operand u_n, and any common factor of the binomial coefficients.

The conditions for ordinary contact are then

$$0=A^n, \qquad 0=A^{n-1}B, \qquad 0=A^{n-1}C.$$

The first of these expresses that A is a point on the surface u_n, the second and third that B and C are on the tangent plane at A.

The further conditions for three-branch contact are

$$0=A^{n-2}B^2, \qquad 0=A^{n-2}C^2, \qquad 0=A^{n-1}D+(n-1)A^{n-2}BC.$$

The first two of these show that B and C are points on the chief tangents at A. If we regard the absolute values of the coordinates of A as given, then it appears from the third equation that B and C may be chosen arbitrarily on the chief tangents and D anywhere in space, the equation giving the relation between the absolute values of their coordinates which determines the particular surface AD−BC=0.

For four-branch contact we have the additional equations,

$$0=A^{n-3}B^3, \qquad 0=A^{n-3}C^3,$$
$$0=2A^{n-2}BD+(n-2)A^{n-3}B^2C,$$
$$0=2A^{n-2}CD+(n-2)A^{n-3}BC^2.$$

The first two of these indicate that B and C lie on the polar cubic of A in regard to the section of u_n by the tangent plane. Now this polar cubic has a node at A, whose tangents are the chief tangents. Each of these lines therefore meets the cubic in three points at A, and cannot have any other point on the curve unless it be itself a part of the cubic. But the points B and C have to lie one on each of the chief tangents. In order, therefore, that all the equations may be satisfied, the polar cubic in question must break up into the two chief tangents and some other line.

This condition may be put into another form. For if we seek the points in which the line AB meets the surface, by substituting the coordinates of $A+\lambda B$ in $u_n=0$, the conditions $A^n u_n=0$, $A^{n-1}Bu_n=0$, $A^{n-2}B^2u_n=0$, $A^{n-3}B^3u_n=0$ indicate that *four* roots of the equation are equal to zero, or that the line meets the surface in four consecutive points at A. We find, therefore, that

Those points of a surface at which a quadric may have four-branched contact are the points at which each chief tangent meets the surface in four consecutive points, or, which is the same thing, *the points whose polar cubic contains the chief tangent.*

The number of these points has been counted by CLEBSCH, Crelle, lxiii. 14 *, and turns out to be

$$n(41n^2-162n+162).$$

A point of this nature being given, one quadric surface having four-branch contact at it may be drawn through another arbitrary point.

The coordinates of the point A being given as to their absolute values, let us substitute for B and C, $A+\lambda B$, $A+\mu C$; where now B and C are fixed points on the chief tangents, whose coordinates are given absolutely. This being so, the following equations are satisfied *ex hypothesi*:—

$$A^n u_n=0, \qquad A^{n-1}Bu_n=0, \quad A^{n-1}Cu_n=0,$$
$$A^{n-2}B^2u_n=0, \quad A^{n-2}C^2u_n=0, \quad A^{n-3}B^3u_n=0, \quad A^{n-3}C^3u_n=0;$$

from which it follows at once, for example, that

$$A^{n-3}(A+\lambda B)^3u_n=0.$$

* The investigation is given by SALMON, Geom. Three Dim. p. 444.

For D also let us substitute νD, where the coordinates of D are now given absolutely. Our three remaining equations are (omitting for shortness the mention of A)

$$0=\nu D+(n-1)\lambda\mu\,.\,BC, \quad (1)$$
$$0=\nu D+(n-2)\lambda\mu\,.\,BC+\nu\lambda\,.\,BD+\tfrac{1}{2}(n-2)\lambda^2\mu\,.\,B^2C, \quad (2)$$
$$0=\nu D+(n-2)\lambda\mu\,.\,BC+\nu\mu\,.\,CD+\tfrac{1}{2}(n-2)\lambda\mu^2\,.\,BC^2; \quad (3)$$

and it remains only to show that these equations determine uniquely λ, μ, ν.

If we subtract (1) from (2) and (3) successively, we obtain

$$0=-\mu\,.\,BC+\nu\,.\,BD+\tfrac{1}{2}(n-2)\lambda\mu\,.\,B^2C, \quad (4)$$
$$0=-\lambda\,.\,BC+\nu\,.\,CD+\tfrac{1}{2}(n-2)\lambda\mu\,.\,BC^2, \quad (5)$$

the values $\lambda=0$, $\mu=0$ not being admissible. But if we substitute in (4) and (5) the value of $\lambda\mu$ derived from (1), we obtain two simple equations which determine the ratios $\lambda:\mu:\nu$; after which the absolute values are uniquely determined from (1).

It is otherwise obvious that if $0=q_2$ be the point-equation to a four-branch quadric, and $0=t_1$ to the tangent plane, $0=q_2+\varrho t_1^2$ will be the equation to a pencil of quadrics having four-branch contact, one of which may be made to pass through an arbitrary point.

Special investigation for n=3.

In the case in which $\mathfrak{u}_n$ is a cubic surface, the points in question are clearly the 135 points of contact of the 45 triple tangent planes, namely, the intersections of the 27 lines. This case may be conveniently studied by means of the representation of that surface on a plane. The plane sections of the surface here correspond to a system of cubics having six common points; the quadric sections therefore to sextics having nodes at these points. The problem is then to draw a sextic curve having six given nodes and a quadruple point elsewhere.

The twenty-seven lines of the cubic are represented by

the 6 fixed points,
the 15 lines joining them, and
the 6 conics each through five of them.

It shall now be proved that the sextic must include two of these; and that for each pair that meet there is a singly infinite number of sextics.

The sextic cannot be a proper curve; for six nodes and a quadruple point are equivalent to twelve nodes, and a proper sextic can have only ten. Nevertheless we may apply a quadric transformation to it, whose principal points are the quadruple point and two of the nodes. The sextic is thus reduced to a quartic, passing 2, 0, 0 times through the principal points respectively and having four other nodes. But a quartic having five nodes must be made up of a conic and two straight lines. Now, if the node at a principal point is the intersection of the two lines, the original sextic was made up of two lines, passing each through two of the six points, and a quartic having nodes at their intersection and the remaining two points and passing through those four. Let a, b, c, d, e, f be the six points, p the intersection of ab, cd; then the sextic is in this case made up of

lines ab, cd,

quartic $p^2e^2f^2abcd$.

If, however, in the transformed figure the node at a principal point was the intersection of a line with the conic, the original sextic was made up of a line, a conic, and a nodal cubic, viz. if p be the intersection of the line ab and the conic $bcdef$, the nodal cubic is p^2acdef.

Now we know that we can draw a singly infinite number of quartics with 3 fixed nodes and 4 fixed points, or of cubics with a fixed node and 5 fixed points. Hence in both these cases the sextic includes two representatives of straight lines in the cubic, together with another curve which may be arbitrarily chosen from a pencil.